Mathematics

Time-dependent carrying capacity in the logistic population growth model

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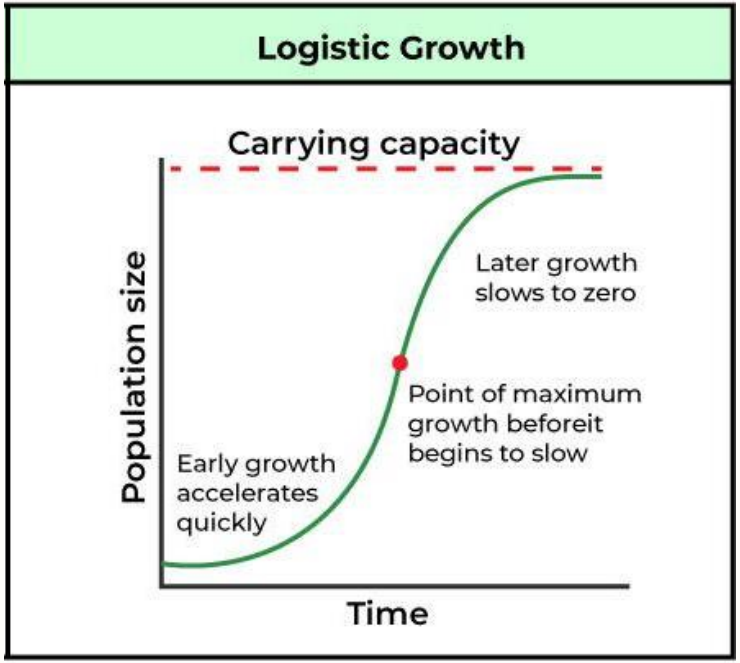
Introduction

In this mathematical research project, I explored the implications of introducing linear and logistically varying time-dependent carrying capacities in the logistic population growth model. Using Python generated graphs, I observed various graphical features such as stationary points, asymptotes and end behavior. By formulating hypotheses, I proved these observations using limits (L'Hôpital's rule), Bernoulli's differential equations and other analytical techniques.



(b)

01

What?

A brief background on logistic population growth model

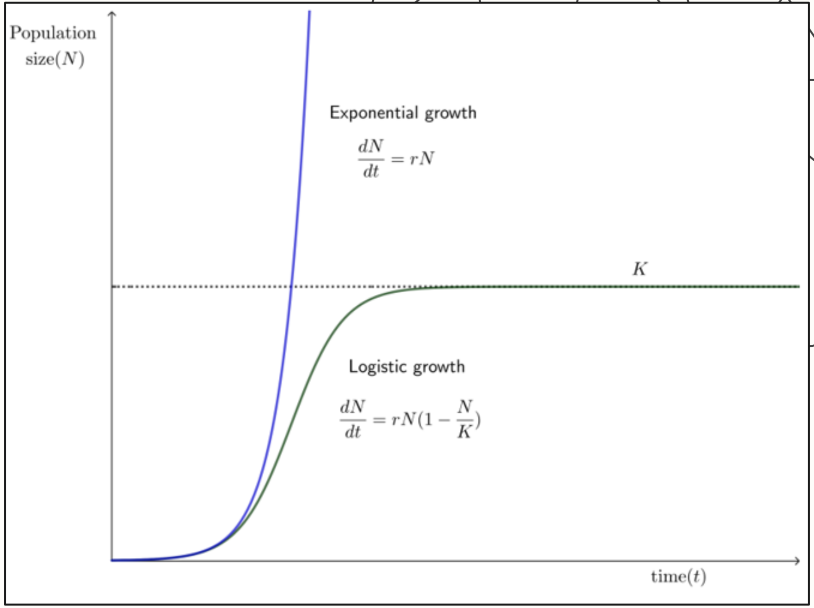
Logistic growth model

-> Defining differential equation:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

-> Basic features:

- Initial behavior similar to exponential growth
- Population keeps on increasing; however, growth rate decreases (as opposed to exponential growth where growth rate is always increasing)
- Approaches a final constant value, K, called the CARRYING CAPACITY (maximum population size of a particular species that a given ecosystem can sustain)

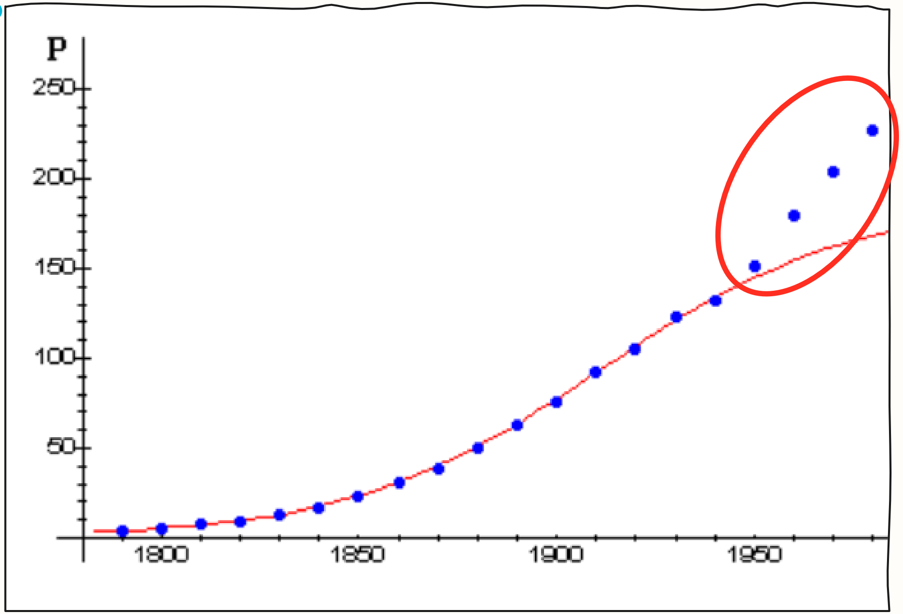


02

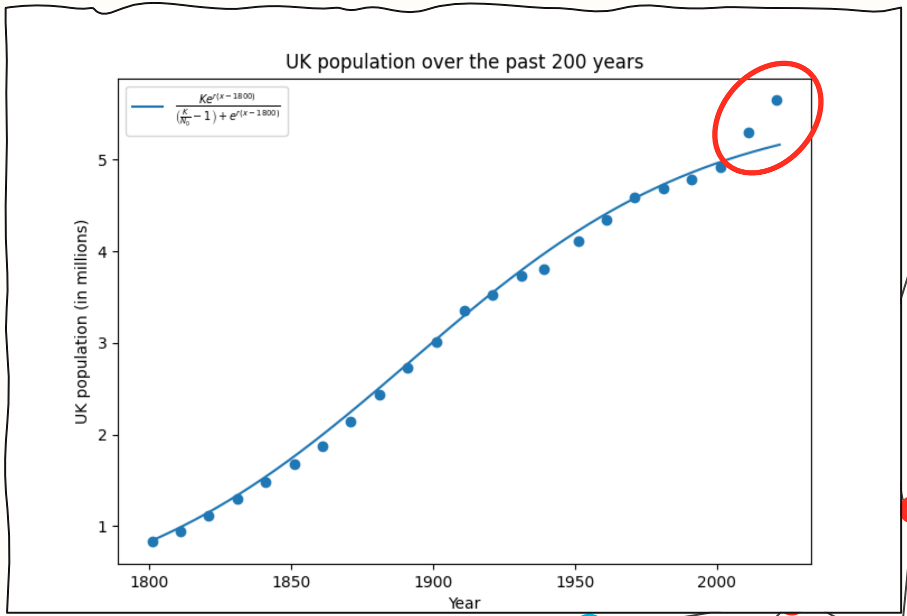
Why?

Rationale behind my intention to explore time-dependent carrying capacity in the logistic population growth model

It only ALMOST works …



Standard logistic best fit applied to US population after Industrial Revolution



Standard logistic best fit applied to UK population after Industrial Revolution

Why a varying carrying capacity (K)?

The standard logistic population growth model assumes constant K, which is a huge assumption since in today's world with climate change, increased research, and technological development, it is improbable that our population will have a fixed maximum size.

Climate change

1

Climate change can make certain areas less habitable and potentially limit the resources available for a growing population.

Technological innovation

2

Innovation like clean energy could further enhance our ability to sustain a growing population.

Why?

Thus, constant K would make little sense and rather varying K must be considered.

Functions used to vary K with time

1) Linear variation i.e.,

$$K(t) = \alpha t + \beta$$

A time-dependent linear carrying capacity implies that the maximum sustainable size of a population in an environment changes linearly with time. There are a few reasons to study such a dependence of the carrying capacity (K) in population growth models:

1. **Climate change and environmental factors:** Altered climate patterns, like temperature increases and extreme weather, stress ecosystems by disrupting resource availability. Pollution and deforestation further reduce resources, diminishing carrying capacity. A negative rate of change of linear carrying capacity allows for modelling the effect of diminishing carrying capacity on the population growth.

2. **Technological development:** The relentless march of innovation, from the Industrial Revolution to the Information Age, has consistently opened up new possibilities and improved living conditions. Today, geoengineering technologies such as stratospheric aerosol injection offer opportunities to continue this trend and allow our planet to support more humans on it. Hence, technological development would increase the carrying capacity of the human population, which can be modelled using a positive rate of change of the linear carrying capacity.

2) Logistic variation i.e.,

$$\frac{dK(t)}{dt} = \alpha_c \left[K(t) - K_i\right]\left[1 - \frac{K(t) - K_i}{K_f}\right]$$

A logistic time-dependent relation of the carrying capacity assists in more accurate representation of the following phenomena:

1. **Introduction of technology:** A logistic carrying capacity is an appropriate fit to model the introduction of a new technology or innovation in a biological environment. This is because, although a new technology will increase efficiency and productivity of the population, these effects are more substantial in the long-term since it does not spread instantly. Initially, the adoption rate tends to be sluggish, primarily because new technology must replace well-established counterparts. However, as time progresses, adoption rate often experiences exponential growth until it encounters physical or other limitations that decelerate the process. [8]

2. **Innovation waves:** Furthermore, it's important to recognize that technological innovations rarely exhibit a uniform temporal distribution. Instead, they tend to emerge in distinct "waves," [8] a pattern evident throughout history, spanning from the Agricultural Revolution and the Scientific Revolution to the Industrial Revolution and, more recently, the Information Revolution. Such phenomenons can be effectively modeled using a population growth model with a logistic carrying capacity representing "waves of development", as we later see in Figure 3 (blue curve).

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How?

What kind of methods and techniques I used to study these models?

Three major techniques

Bernouilli differential equations

All the models were defined using differential equations which took this specific form

Computational plot generation (using Python)

Since solutions obtained were not closed-form, Python plot methods were used to create graphs

Mathematical analysis

Limits, differential, L'Hôpital's rule, and other tools were used to analyze the graphs

Important method used (similar to actual research)

A very important feature of this work:

1. Graphs were plotted using Python
2. Graphical features that were observed were used to formulate conjectures
3. These conjectures were then proven rigorously in full generality.

04

So?

A summary of important results obtained and their implications

Major results:

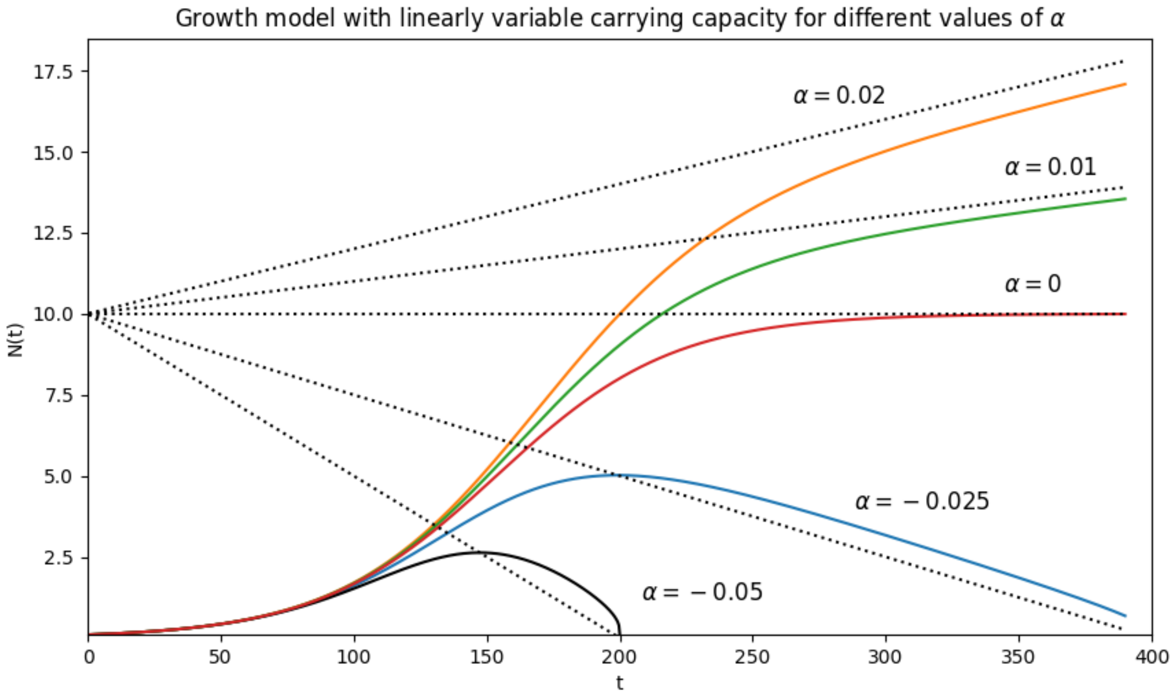
- Different models exhibit different features, such as points of maxima, asymptotic end behavior, and more

- Growth rate is the major factor which affects the behavior of the newly defined population growth models with varying carrying capacities

- Results observed for specific parameters in a model can be fully generalized and proven using analytical tools such as derivatives, limits, L'Hôpital's rule, etc.

Model 1 - Linearly varying carrying capacity

$$K(t) = \alpha t + \beta \;\Rightarrow\; N(t) = \frac{e^{rt}}{r\int \frac{e^{rt}}{\alpha t+\beta}\,dt}$$



Model 1 - Theorems

1. For $\alpha > 0$, as time increases without bound, $N(t)$ approaches $K(t)$.

2. For $\alpha < 0$, $K(t)$ intersects $N(t)$ at the point of maximum of $N(t)$.

A snippet of one of the proofs

Theorem 1.

In a logistic growth model with a time-dependent linear carrying capacity, when the rate of change of carrying capacity is positive i.e., $\alpha > 0$, as time t increases without bound, $N(t)$ approaches $K(t)$.

Proof

First, let us represent the statement in formal limit notation:

$$\lim_{t\to\infty} \frac{N(t)}{K(t)} = 1.$$

This is justified since when two functions would approach each other, their ratio should tend to 1.

Next, knowing that $\lim_{t\to\infty} K(t) = \lim_{t\to\infty}(\alpha t + \beta) \neq 0$ since $\alpha \neq 0$, we can use the quotient rule of limits [3] to write $\lim_{t\to\infty} \frac{N(t)}{K(t)}$ as follows:

$$\lim_{t\to\infty} \frac{N(t)}{K(t)} = \frac{\lim_{t\to\infty} N(t)}{\lim_{t\to\infty} K(t)}. \tag{13}$$

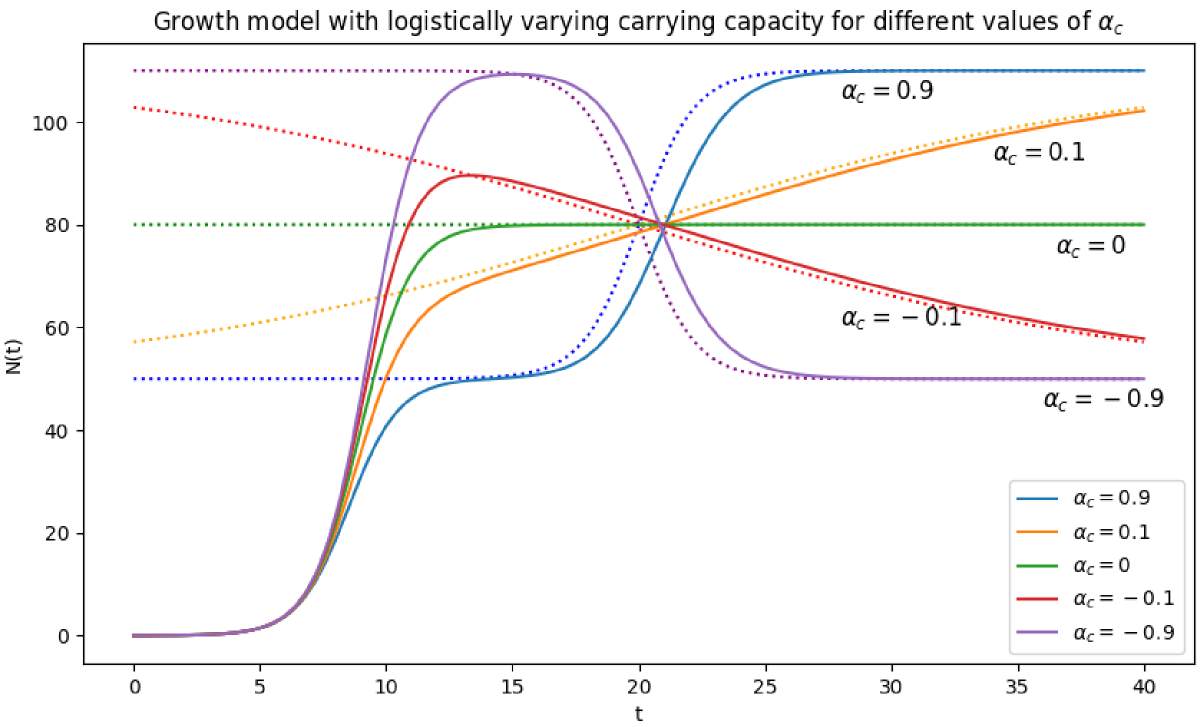
Let us solve the limit in the numerator, $\lim_{t\to\infty} N(t)$, using the expression derived for $N(t)$ in equation (12) on page 9.

$$\begin{aligned} \lim_{t\to\infty} N(t) &= \lim_{t\to\infty} \frac{e^{rt}}{r\int \frac{e^{rt}}{\alpha t+\beta}\,dt}, \\ &= \frac{1}{r} \cdot \lim_{t\to\infty} \frac{e^{rt}}{\int \frac{e^{rt}}{\alpha t+\beta}\,dt}. \end{aligned}$$

Now, knowing that $r > 0$ from the assumption made in section 2.1 on page 3, when $t \to \infty$, $e^{rt} \to \infty$.

Model 2 - Logistically varying carrying capacity

$$K(t) = K_i + \frac{K_f}{1 + e^{-\alpha_c(t-t_0)}} \quad \Rightarrow \quad N(t) = \frac{e^{rt}}{r\int \frac{e^{rt}}{K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}}\,dt}$$



Model 2 - Theorems

1. For $\alpha_c < 0$, $K(t)$ intersects $N(t)$ at the point of maximum of $N(t)$.

2. $\lim_{t\to\infty} N(t) = \lim_{t\to\infty}(K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}) = \lim_{t\to\infty} K(t)$

A snippet of one of the proofs

Theorem 3.

In a logistic growth model with a time-dependent logistic carrying capacity, when the intrinsic growth rate of the carrying capacity is negative i.e., $\alpha_c < 0$, $K(t)$ intersects $N(t)$ at the point of maximum of $N(t)$.

Proof

The above theorem can be proven through the following steps:

(a) Let us assume that the point of intersection of $K(t)$ and $N(t)$ is $(t_j, N(t_j))$. To show that this is a local maximum of $N(t)$, we need to first prove that it is a stationary point i.e., $N'(t_j) = 0$ and then employing the second derivative test, we must show that $N''(t_j) < 0$.

(b) Let us first prove $N'(t_j) = 0$.

At $t = t_j$, $K(t)$ and $N(t)$ intersect each other and hence $K(t_j) = K_i + \frac{K_f}{1+e^{-\alpha_c(t_j - t_0)}} = N(t_j)$. Now we can find $N'(t_j)$ by substituting $t = t_j$ in the differential equation which defines this model i.e., equation (17) on page 19.

$$N'(t_j) = \frac{dN(t_j)}{dt} = rN(t_j)\left[1 - \frac{N(t_j)}{K_i + \frac{K_f}{1+e^{-\alpha_c(t_j - t_0)}}}\right].$$

Since $K_i + \frac{K_f}{1+e^{-\alpha_c(t_j - t_0)}} = N(t_j)$,

$$\begin{aligned} N'(t_j) &= rN(t_j)\left[1 - \frac{N(t_j)}{N(t_j)}\right], \\ &= rN(t_j)(1-1), \\ &= 0. \end{aligned}$$

Hence, we have shown that $N'(t_j) = 0$.

$$\begin{aligned} \frac{d}{dt}\left(\frac{dN}{dt}\right) = \frac{d^2N}{dt^2} &= \frac{d}{dt}\left[rN\left(1 - \frac{N}{K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}}\right)\right], \\ &= r\frac{d}{dt}\left[N - \frac{N^2}{K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}}\right], \\ &= r\left[\frac{dN}{dt} - \frac{d}{dt}\left(\frac{N^2}{K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}}\right)\right], \\ &= r\left[\frac{dN}{dt} - \frac{\left[K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right]\frac{d(N^2)}{dt} - N^2\left[\frac{d}{dt}\left(K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right)\right]}{\left[K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right]^2}\right], \\ &= r\left[\frac{dN}{dt} - \frac{\left[K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right]\cdot 2N\frac{dN}{dt} - N^2\cdot\left[\frac{d}{dt}\left(K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right)\right]}{\left[K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right]^2}\right]. \end{aligned}$$

Now, remembering K_i, K_f, t_0 and α_c are constants, we can evaluate $\frac{d}{dt}(K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}})$ as follows:

$$\begin{aligned} \frac{d}{dt}\left(K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right) &= K_f\cdot\frac{-1}{(1+e^{-\alpha_c(t-t_0)})^2}\cdot e^{-\alpha_c(t-t_0)}\cdot(-\alpha_c), \\ &= \alpha_c\,\frac{K_f\,e^{-\alpha_c(t-t_0)}}{(1+e^{-\alpha_c(t-t_0)})^2}. \end{aligned}$$

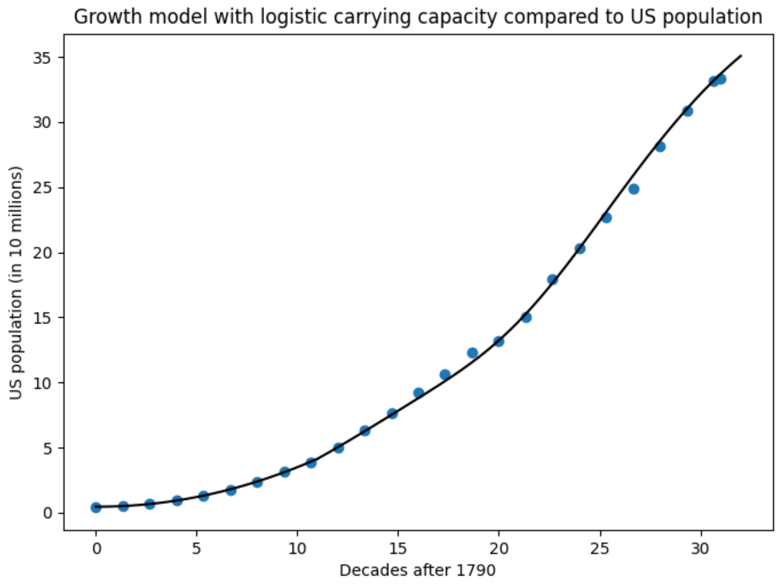
Substituting this value in the expression for $\frac{d^2N}{dt^2}$,

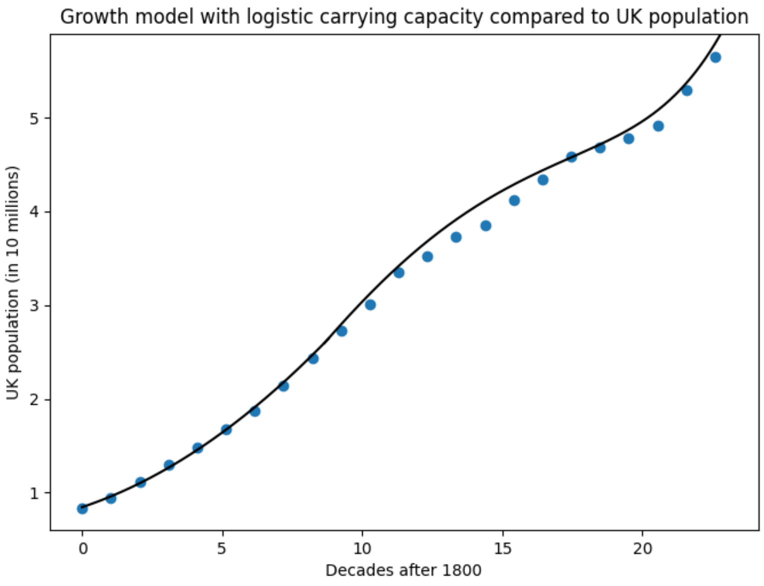
$$\begin{aligned} \frac{d^2N}{dt^2} &= r\left[\frac{dN}{dt} - \frac{\left[K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right]\cdot 2N\frac{dN}{dt} - N^2\cdot\left[\frac{d}{dt}\left(K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right)\right]}{\left[K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right]^2}\right], \\ &= r\left[\frac{dN}{dt} - \frac{\left[K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right]\cdot 2N\frac{dN}{dt} - N^2\cdot\left[\alpha_c\frac{K_f\,e^{-\alpha_c(t-t_0)}}{(1+e^{-\alpha_c(t-t_0)})^2}\right]}{\left[K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right]^2}\right], \\ &= r\frac{dN}{dt} - \frac{2rN\frac{dN}{dt}}{K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}} + \frac{rN^2\cdot\alpha_c\frac{K_f\,e^{-\alpha_c(t-t_0)}}{(1+e^{-\alpha_c(t-t_0)})^2}}{\left[K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right]^2}. \end{aligned}$$

Thus, we have obtained a general expression for the second derivative of $N(t)$,

$$N''(t) = rN'(t) - \frac{2rN(t)N'(t)}{K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}} + \frac{rN^2(t)\cdot\alpha_c\frac{K_f\,e^{-\alpha_c(t-t_0)}}{(1+e^{-\alpha_c(t-t_0)})^2}}{\left[K_i + \frac{K_f}{1+e^{-\alpha_c(t-t_0)}}\right]^2}.$$

Aligning new model with population data





Conclusion and further research

Overall, the new model provides a deeper and more realistic understanding of human population growth which is also in sync with how technology and climate change affect our species' size.

Further research:

1. Factoring in fertility and mortality rates and other variables
2. Investigating a population-dependent carrying capacity with a variable growth rate
3. Exploring models with different mathematical functions for carrying capacity, such as sinusoidal, logarithmic, hyperbolic, etc.

Thanks!

Keep growing!